Mid-Term Exam ISI Bangalore February 2016

B Math Algebra II 100 Points

Notes.
(a) Justify all your steps.
(b) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
(c) By default, F denotes a field.

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1. [15 points] Prove that centre of the group $\mathrm{GL}_n(F)$ is the set of all scalar matrices in $\mathrm{GL}_n(F)$.

2. [15 points] Let A be an $n \times n$ matrix with integer entries such that all the diagonal entries are even numbers and the rest are odd. If n is an even number, prove that A is invertible over $\mathbb{Q}$.

3. [15 points] Let $V \xrightarrow{T} W$ be a linear transformation. If W and $\ker(T)$ are finite-dimensional, then prove that V is finite dimensional.

4. [15 points] Find the number of 3-dimensional subspaces in $\mathbb{F}_p^n$. Justify your answer.

5. [15 points] Given any two nonzero vectors $v, w \in F^n$, prove that there exists an invertible $n \times n$ matrix A such that $Av = w$.

6. [15 points] Let A, B be $m \times n$ matrices over F. Prove that $\operatorname{rank}(A + B) \leq \operatorname{rank}(A) + \operatorname{rank}(B)$.

7. [10 points] Find a basis of the solution space for $A\vec{x} = \vec{0}$ where A and $\vec{x}$ are given by

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 7 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 & 10 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

(Either check the basis property of your solution directly, or quote suitable results proved in class.)